Commutative partially ordered monoids will be referred to as uninorms in this paper. The aim of this paper is to investigate involutive uninorms:

**Definition 1** \( U = \langle X, \ast \circ, \leq, t, f \rangle \) is called an involutive FL\(e\)-algebra if

1. \( C = \langle X, \leq \rangle \) is a poset,
2. \( \ast \) is a uninorm over \( C \) with neutral element \( t \),
3. for every \( x \in X \), \( x \rightarrow \ast f = \max \{ z \in X \mid x \ast z \leq f \} \) exists, and
4. for every \( x \in X \), we have \( (x \rightarrow \ast f) \rightarrow \ast f = x \).

We will call \( \ast \) an involutive uninorm. Our main question is the following: in an involutive FL\(e\)-algebra, how far its uninorm (or its algebraic structure, in general) is determined by its “local behavior”, i.e., its underlying t-norm and t-conorm. An answer to this question is presented in Section 4, for a particular case on \([0, 1]\) with \( t = f \), which will illustrate our background idea. It says that the uninorm is determined uniquely by any of them, i.e., either by the t-norm or by the t-conorm. In fact, the t-norm and the t-conorm are determined by each other, in this case. Then, a natural question is how far we can extend this, and when the uninorm is determined uniquely?

The main goal of the present paper is to give an answer to this question. The first attempts in this direction are made in Section 3: uniqueness is guaranteed (Corollary 4) and moreover, the uninorm is represented by the twin-rotation construction (Corollary 6) whenever the algebra is conic. To have a closer look at the situation, in Section 3 we consider involutive FL\(e\)-algebras which are finite and linearly ordered. As a byproduct it follows that the logic \( \text{IUL} \) extended by the axiom \( t \leftrightarrow f \) does not have the finite model property (Corollary 4).

### 2 Preliminaries

It is not difficult to see that every involutive uninorm is residuated (see [15]) and hence \( \ast \) is isotone (see [10]). Therefore, \( \prime : X \to X \) given by

\[
x' = x \rightarrow \ast f
\]

is an order-reversing involution. If \( C \) is linearly ordered, we call \( U \) an involutive FL\(e\)-chain. \( U \) is called finite if \( X \) is a finite set. \( U \) is called bounded if \( X \) has top \( \top \) and bottom \( \bot \) elements. Observe that the notion of bounded involutive FL\(e\)-algebras with \( f = \bot \) coincides with the notion of Girard monoids, cf. [10, 13].

A partially ordered monoid is called integral (resp. dually integral) if the poset has its greatest (resp. least) element and it coincides with the neutral element of the monoid. Uninorms which are integral (resp. dually integral) will be referred to as t-norms (resp. t-conorms). For any uninorm \( \ast \) with neutral element \( t \) on the poset \( \langle X, \leq \rangle \) define its positive and the negative cones by

\[
X^+ = \{ x \in X \mid x \geq t \} \quad \text{and} \quad X^- = \{ x \in X \mid x \leq t \},
\]

respectively. The algebra, and as well \( \ast \) is called conic if every element of \( X \) is comparable with \( t \), that is, if \( X = X^+ \cup X^- \). A moment’s reflection shows that \( \ast \) restricted to \( X^+ \) (resp. \( X^- \)) is a t-conorm (resp. t-norm), call them the underlying t-conorm and t-norm of \( \ast \), respectively. Thus uninorms have a block-like structure; they have an underlying t-norm and t-conorm, that is, a t-norm and a t-conorm act on \( X^+ \) and on \( X^- \), respectively. Now two questions arise naturally.