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Fuzzy Set Theory

Preferences, Games and Decisions

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Abstracts

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János Fodor, Erich Peter Klement, Marc Roubens
Editors

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Since their inception in 1979 the Linz Seminars on Fuzzy Sets have emphasized the development of mathematical aspects of fuzzy sets by bringing together researchers in fuzzy sets and established mathematicians whose work outside the fuzzy setting can provide direction for further research. The seminar is deliberately kept small and intimate so that informal critical discussion remains central. There are no parallel sessions and during the week there are several round tables to discuss open problems and promising directions for further work.

LINZ 2006 will be already the 27th seminar carrying on this tradition, will be devoted to the mathematical aspects of "Preferences, Games and Decisions". As usual, the aim of the Seminar is an intermediate and interactive exchange of surveys and recent results.

János Fodor
Erich Peter Klement
Marc Roubens

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Fuzzy Truth, Partial Truth, and Games

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We begin with the category **Sets²** as defined in [1] (see esp. p. 25 and pp. 35-36). This is the category whose objects are functions from one set to another, denoted by $\sigma: X \to X'$. The arrows then are commutative squares between the objects (the σ's). For **Sets²**, the subobject classifier Ω (the set of truth values) becomes the arrow $\sigma: \{0, 1, 2\} \to \{0, 1\}$ which sends 0 to 0 (true, σ0), 1 to 0 (true, σ1) and 2 to 1 (false, σ2).To understand this, note that a subobject of an object $\sigma: X \to X'$ would be an arrow $\sigma: S \to S'$ where $S \subseteq X$, $S' \subseteq X'$ and $\sigma(S) \subseteq S'$. The characteristic function then maps $X(x) \to X'(x)$ to σ0 if $x \in S$, to σ1 if $x \notin S$ but $\sigma(x) \in X'$, and to σ2 if $x \notin S$ and $\sigma(x) \notin X'$. Thus, as is noted in [1], the characteristic function tells us whether "x is in S" is true always, just at 1, or never. The question is exactly how do we come to this particular subobject classifier? Why, for instance, could we not simply use (say) $\rho: \{0, 1\} \xrightarrow{id} \{0, 1\}$ by analogy to the Ω for sets ({0, 1})?

The obvious answer is that using ρ as the characteristic function would conflate the status of the elements in X that come to be in S when σ is applied to X (the elements true "just at 1") with the status of the elements that do not come to be so (the elements that are true from the beginning). But why should this matter? Because then there can be no suitable characteristic function from $X \to X'$ to $\{0, 1\} \xrightarrow{id} \{0, 1\}$for a subobject whose underlying morphism takes some x not in S to S'. We need the characteristic function to take all members of S to 0 (true) and all members of S' to 0 (also true) so that the "truth function" square will commute. But this will not be the case: one side of the square will take $\sigma: S \to S'$ to 1 to $0 \to 0$ (true) only for $x \in S$, while the other side will take the subset $\sigma: S \to S'$ to $\sigma: X \to X'$ which will need to be mapped to $\{0, 1\} \xrightarrow{id} \{0, 1\}$. All we can do here is map S to 0 and S' to 0, but this will "leave out" $x \to x'$ ($x \notin S$, $x' \in S'$) since it would require a morphism in Ω which takes 1 to 0, and we have not provided one. Thus the structure of **S²** itself dictates the Ω chosen in [1]. Remember also in this regard that the characteristic morphism (function) ϕ_S must be a component of a pullback, that is, that the monomorphism is the pullback of *true* ($1 \to \Omega$) along ϕ_S. In **Sets**, for instance, it is easy to see that this is correct. Choose a monomorphism (subset) S of a set X and call the characteristic function of X for subset S X_S. Then clearly all other commutative squares to Ω through X_S will factor through S since their origins will be subobjects (subsets) of S. It is not possible in **Sets** for $S' \to 1$ (true)$\to \Omega$ to commute with $S' \to X$ (X_S)$\to \Omega$ if S' is not a subset of S.

It is useful to examine another category, a sort of recursive version of **S²**, which requires an infinite number of truth values, i.e., an infinite Ω which is nonetheless unique. Consider the category
ε
of endomaps of sets, from [2], whose objects are single sets equipped with an endomap and denoted (X; α). By reasoning very similar to that described above for **S²**, it is apparent that for any subobject $(S, \alpha|S)$ of (X; α) (i.e., a subset of X closed under α), an element in X may be taken to S by one application of α (x is in S), by two (i.e., $\alpha \circ \alpha$), by three ($\alpha \circ \alpha \circ \alpha$), by any number of applications ($\alpha \circ \alpha \circ ... \circ \alpha$) or never. And each of these iterations creates another subobject of (X; α) with the same subset S, so all of these must be classified Thus, an element of X must be classified, relative to a particular subobject, by the number of iterations of the endomorphisn required to get it into the

subset of the subobject, and the unique Ω therefore is (N; γ), where N is the set of natural numbers and γ takes the natural number n to n-1. Thus $x \in X$ is mapped to 0 in Ω if it is an "original" member of S in X, to 1 if it becomes a member (i.e., a generalized element - see, e.g., [2], pp. 8-9) of S after one iteration of α, and so on. An x in X mapped to ∞ never becomes a member of S (no matter how many times α is iterated), so ∞ represents completely false. There are, in other words, an infinite number of truth values for the category $\boldsymbol{\varepsilon}$, yet Ω is nevertheless unique and motivated entirely by the structure of the category. Following [2], we refer to the degrees of truth assigned to elements of $\mathbf{S}^2$ or $\boldsymbol{\varepsilon}$ as "partial truths."

Before turning specifically to fuzzy sets and their subobject classifiers, we provide some background to make sure all preliminaries are clear. First of all, by a presheaf on a category **C** we mean a *contravariant* functor from **C** to **Sets** (i.e., $\mathbf{Sets}^{\mathbf{C}^{op}}$, as in [1]) and not a covariant functor from **C** to **Sets** as in [2]. Thus, generally speaking, any reference to [2] will require an implied *mutatis mutandis*. Since we will want to think of fuzzy sets as functors (presheaves, members of $\mathbf{Sets}^{\mathbf{C}^{op}}$), it is important to keep in mind that in $\mathbf{Sets}^{\mathbf{C}^{op}}$, all subfunctors are subobjects and conversely (see [1], p. 36). By straightforward application of the Yoneda lemma (as in [1], p. 37), the set of "truth values" for an object C of **C** in a presheaf must be (isomorphic to) the set of subfunctors of $\mathrm{Hom}_{\mathbf{C}}(-, C)$. Of course, this is $\boldsymbol{\Omega}(C)$ and will be sufficient to classify only $\mathrm{Hom}_{\mathbf{C}}(-, C)$; to classify subobjects of objects in $\mathbf{Sets}^{\mathbf{C}^{op}}$ in the general case we need $\boldsymbol{\Omega}(.)$ for each object in **C**.

A helpful way to view the elements of $\boldsymbol{\Omega}(C)$ is as sieves (ibid.). A sieve S on C in **C** is a set of arrows in **C** with codomain C s.t. if f is in S and $f \bullet h$ is defined then $f \bullet h$ is also in S. For a poset, a sieve is simply a set of elements $B \leq C$ s.t. if $A \leq B \in S$ then $A \in S$. In any locally small category, the sieves on C are the same as the subfunctors of $\mathrm{Hom}_{\mathbf{C}}(-, C)$ ([1], p. 38). Now take any presheaf $\mathbf{Sets}^{\mathbf{C}^{op}}$, any member functor P and subfunctor Q (not necessarily hom-functors), and any morphism $f: A \to C$ in **C**. Then f determines a function $P(f): P(C) \to P(A)$ in **Sets**. For any given x in P(C), P(f) may take x into Q(A) or it may not, and the set $\{f \mid x \bullet f \in Q(\mathrm{dom}(f))\}$ is a sieve on C where f ranges over all morphisms with codomain C. This sieve is the "set of all those paths f to C which translate the element x of P(C) into the subfunctor Q." ([1], p. 39) It is also, therefore, the "truth value" of x. If $x \in Q(C)$, then this operation will yield the maximal sieve on C; if $x \notin Q(C)$, then this operation will yield some other sieve on C which is not maximal and which may be the empty sieve.

We now turn specifically to subobject classifiers for fuzzy sets. Let us begin with the subobject classifier for the categories **Set H**and **Mod H** as described by Wyler in [3] (p. 255). Recall that an H-set is a pair $(|A|, \delta_A)$, where H is a complete Heyting algebra, |A| is a set and δ_A is a symmetric and transitive mapping from |A| x |A| to H. Degree of membership in an H-set is given by $\varepsilon_A = \delta_A(x, x)$. Thus H-fuzzy sets are (can be) totally fuzzy sets (both membership and equality are fuzzy). An H-valued fuzzy relation $f: A \to B$ is an extensional mapping $f: |A| \times |B| \to H$. The category **Set H** is defined, then, to be the category with H-sets as objects and H-valued relations as morphisms, and **Mod H** to be the category with H-sets as objects and H-valued relations as morphisms, but only those induced mappings of the underlying sets. We shall focus on **Set H** here.

The easiest way to describe subobject classification in **Set H** is in terms of H-subset structures. For a given H-set A in **Set H**, an H-subset structure of A is a mapping $\alpha : A \to H$ where $\alpha(x) \leq \varepsilon_A(x)$ and $\alpha(x) \wedge \delta_A(x, x') \leq \alpha(x')\ \forall x, x'$ in |A| ([3], p. 249). An H-subset A_α of A, then, with the given H-subset structure α, is the H-set $(|A|, \delta_\alpha)$, where $\delta_\alpha(x, x') = \alpha(x) \wedge \delta_A(x, x')$. An injective morphism can easily be constructed which takes A_α to A. Now take the set of truth values Ω to be the elements of the complete Heyting algebra H. Then the characteristic morphism ch j_α for A_α is the morphism induced by α, that is, ch j_α (x, a) = $\varepsilon_{A\alpha}(x) \wedge \delta_\Omega(\alpha(x), a)$. This means that for each x in |A|, its truth value is a fuzzy set on H whose membership value at each a in H is either $\varepsilon_{A\alpha}(x)$ or a (cf. [3], p. 255; this simplification follows in particular from the fact that all relations in **Set H** must be total, i.e., $\vee_{y\in|B|}f(x, y) = \varepsilon_A(x)$ ([3], p. 244)). The min operator and the definition of an H-subset structure guarantee that the fuzzy truth value of any H-subset A_α of A will be ≤ the fuzzy truth value of the H-set A taken as a subset of itself. These truth values are, of course, equivalent to sieves in the functor category $\mathbf{Sets}^{\mathbf{Hop}}$. Note that this is "sheaf-theoretic" truth; truth here is not "partial" in the sense of truth in $\mathbf{S}^2$ or $\boldsymbol{\varepsilon}$ as described above. We shall refer to this kind of truth here as "fuzzy truth" to distinguish it from the notion of partial truth just mentioned. A crucial difference here is that fuzzy truth does not seem to reflect the idea of "stages of truth" or temporal truth ([2], esp. Section 3.3).

With this background, we now ask what might be the relevance of partial truth and fuzzy truth to the areas of decision theory and game theory. This issue has been raised in a general way by Voinov in [4]. He observes that in many decision making contexts, strict "numerifications" of similarity data (e.g., projections onto metric spaces) are not appropriate or illuminating. "Nearness" relations (basically, suitable subsets of the cartesian product) , on the other hand, may capture much more closely the actual similarities of the system under study, but they are likely to have more rudimentary mathematical structure (say the structure of a pre-uniformity). Another aspect of decision making , Voinov points out, is the notion of cognitive spaces or "regions of evidence" relative to which similarities and other kinds of evidence may be interpreted. These spaces are usually explained "modally," i.e., as manifestations of possible worlds in modal logics, including fuzzy modal logics. The difficulty here, in Voinov's opinion, is that once again the construction of possible worlds and making choices among them requires arbitrary application of numerical techniques.

Voinov suggests that the correct level of mathematical generality for both "connectivity" (possible world choices) and "similarity grade" relationships is that of a topos. More than that, he notes that this incorporates both notions into a single mathematical framework, since the object structure of the topos (the topology) provides a framework for possible worlds and their interrelationships, while the subobjects and subobject classification provide a basis for similarity grades. In other words, in a topos, there is no need to provide separate formalisms for the set of possible worlds and the set of truth values.

Now let us consider briefly fuzzy games, in particular the variety known as fuzzy moves [5]. The (crisp) theory of moves (TOM) was originally developed by S. J. Brams [6]; a nice description of the basic framework may be found in [7]. In "basic" TOM, each game is 2x2, each player is assigned two actions (strategies), and at her turn, a player may move (go to the next strategy) or not move (retain the current strategy). An equilibrium, known as a non-myopic equilibrium (NME) is achieved when a player decides not to move. Players are given a starting state, and are expected to look ahead as far as necessary to determine their next move. This means, as Ghosh and Sen [7] point out, that it is not actually necessary to play the game to determine the NME. It is important to note that only relative payoffs need be used, and also to note that while access to complete information is assumed in the original formulation, it is possible (see [7]) to allow the players to learn their opponent's preferences dynamically. In [6], Brams exhaustively enumerates all 78 possible 2x2 games, and so 2x2 TOM games are usually referred to by their position in Brams' list, e.g. "game 23". It is apparent that a 2x2 TOM can be thought of as a member of the category $\boldsymbol{\varepsilon}$ of endomaps of sets described above.

In [5], Kandel and Zhang suggest that TOM games can be made more complete by adding a fuzzy component. This component consists of an assignment of a value in [0, 1] for each player to each payoff in the 2x2 game; the assigned value represents the overall desirability of the state for that player. This permits a more comprehensive and (according to [5]) more realistic situation in which players attempt to maximize both the order (local) payoff and the fuzzy (global) payoff for a game. The payoff for a given state (a^{-}_{ij}), then, is a "transformation function" [5] of the ordinal (now fuzzified) payoffs of the players and the global (fuzzy) goal α for player A or β for player B, i.e., for player A, $a^{-}_{ij} = F(a_{ij}, b_{ij}, \alpha)$ where i, j ε {1, 2}. The strategies at each row and column intersection in the crisp 2x2 game (the pair of ordinals (a_{ij}, b_{ij}) for row i and column j are transformed in the fuzzy game, then, to (a^{-}_{ij}, b^{-}_{ij}), and the decision to move or not to move is made on the basis of these new (fuzzy) values.

It is apparent that totally fuzzy sets, and hence the category **Set H**, are appropriate models for and generalizations of the theory of fuzzy moves (TFM) of [5] just described. For player A, the membership value of a state represents its global goal, i.e., $\mu(a_{ij}) = \varepsilon(a_{ij}) = \delta(a_{ij}, a_{ij}) = \alpha$ and $\mu(b_{ij}) = \delta(b_{ij}, b_{ij}) = \beta$, while $\delta(a_{ij}, a_{kl})$ represents the local payoff or preference of a_{ij} relative to a_{kl} expressed in terms of equality, i.e., if $\delta(a_{ij}, a_{kl})$ is small, then state a_{ij} and state a_{kl} are far apart from each other in (ordinal) value, and vice versa. The original (crisp) ordinal ranks may then be recovered from the membership values and the equalities taken together. Given two H-sets A and B, the global payoffs a^{-}_{ij} are now given by morphisms in **Set H**, from A to B for the A component of the payoff, and from B to A for the B component. The required properties of a morphism in **Set H** ensures that these global payoffs will be functionally related to the global goals and the local payoffs. Thus, for instance, the requirement that morphisms be total guarantees that the value in H of any f: A $\rightarrow$ B will be $\leq \varepsilon(a)$ for every a in |A| (see [3], pp. 244-245).

It should be emphasized that totally fuzzy sets in **Set H** constitute both a model and a generalization of TFM, or at least of the normal TFM game with normal global goal (see [5], p. 165). Thus, for instance, TFM allows just a single global goal, whereas (obviously) each state (member of |A|) in A can have a different global goal. On the other hand, using sets and morphisms in **Set H** for TFM imposes certain constraints on various aspects of TFM which are not necessarily inherent in the formulation of [5]. One such constraint, viz., that morphisms be total, was mentioned in the previous paragraph. Similarly, the requirement of extensionality ([3], p. 243) guarantees that the value in H of any f: A $\rightarrow$ B will always be $\leq \varepsilon(b)$. These constraints seem reasonable and natural, but may yield a somewhat different set of global preferences than the original TFM. Finally, it should be noted that given Proposition 74.6 and Corollary 74.6.1 of [3] (p. 251), there is an important connection between H-subsets in **Set H** and morphisms in **Set H**. This suggests an important truth-functional connection between "allowed" global payoffs and the underlying local payoffs and global goals. This in turn may lead to a statement of conditions for stronger kinds of equilibria than NME in TFM games, as well as a clear statement of the way fuzzy truth (in the sense the term was used above) may subsume the partial truth of [2].

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Fuzzy Transitive Relations

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Max-min transitivity faces a lot of criticism (see [1, 2, 3]). This criticism seems to be fair particularly because the existing definition decides whether a relation is transitive or not? In this note we are more interested in knowing how much transitive it is? This is what one should call fuzzy transitivity. This transitivity can be extended from crisp transitivity by fuzzifying the operators involved.

If one looks at the definition of crisp transitivity: A binary relation R on a set X is transitive if and only if $(\forall(x, y, z) \in X^3)((xRy \wedge yRz) \Rightarrow xRz)$. The two operators used are conjunction and implication, both of which have been extended to their fuzzy counterparts by this time (implicators were not converted to fuzzy ones at the time when Zadeh [4] defined his max-min transitivity). To get fuzzy transitivity one should fuzzify the operators used.

DEFINITION 1: (*pointwise*) Let R be a fuzzy relation on X. A *transitivity function* $Tr: X \times X \rightarrow [0,1]$ is defined by

$$\mathrm{Tr}(x, z) = \inf_{y \in Y} \mathrm{I}(\mathrm{T}(\mathrm{R}(x, y), \mathrm{R}(y, z)), \mathrm{R}(x, z)).$$

Where I is the implicator corresponding to the t-norm T.

Transitivity function Tr, so defined is a pointwise defined function, which assigns a degree of transitivity to the relation at each point of $X \times X$. According to Definition 1, the given transitive fuzzy relation have different degrees of transitivity at different pairs of points. This fact when incorporated into similarity or indistinguishbility may be interpreted as “different pairs of points may be less or more similar under the same relation”. Now, the problem is to decide how much transitive a fuzzy relation is? The most natural answer seems to take the inf over all the points $x, y, z \in X$.

DEFINITION 2: Let R be a fuzzy relation on X. *Fuzzy transitivity of R* is a function defined as

$$Tr(R) = \inf_{(x,y,z) \in X^3} \mathrm{I}(\mathrm{T}(\mathrm{R}(x, y), \mathrm{R}(y, z)), \mathrm{R}(x, z)) = \inf_{(x,z) \in X^2} \mathrm{Tr}(x,z).$$

Where I is the implicator corresponding to the t-norm T.

REMARK 3: A fuzzy relation R will be called *non-transitive* if $Tr(R) = 0$, and it will be called *strongly transitive* if $Tr(R) = 1$. The first case is basically dealing with crisp non-transitive relations as a case of the fuzzy one, and in the later case *R* is going to have a transitivity value 1 at each of it's triplet of points which means crisp transitivity

REMARK 4: Where is Zadeh's definition of transitivity placed in this situation?

Before going onwards let us name for the sake of convenience in calculation:

a = R(x, y), b = R(y, z), and c = R(y, z).

For any $x, y, z \in X$.

Tr (x, y, z) = I(T(a, b), c),

(where *a, b, c* are points dependent). Zadeh [4] assumed the first variable of the implicator to be smaller than the second one i.e. T (a,b) ≤ c which leads to a value 1 for almost all the implicators we prefer to use. So while working with these implicators Zadeh's definition is the second case stated in remark 3 i.e. the crisp transitivity. Some of the implicators may give values other than 1 in case of taking max-min transivity valid. The study of such situations is in progress.

REMARK 5: What happens to the so defined equivalence relation?

Next step after having defined a fuzzy transitivity, is fuzzy equivalence relation. That is equivalence up to a certain degree. That should be the exact picture of indistinguishbility i.e.; How indistinguishable two points are with respect to the pseudo-metrics associated with equivalence relation? If they are totally indistinguishable than an interesting fact is that similarity so defined conforms following three properties:

1. If *R* is a similarity relation then $\forall$ $(x,y,z) \in X^3$, at least two of the degrees *R(x,y), R(x,z)* and *R(y,z)* are equal.
2. *R* is a similarity relation if and only if for all $\alpha \in]0,1]$ the α-cut R_α is an equivalence relation.
3. *R* is a similarity relation if and only if complement of *R* is a *[0,1]*–valued pseudo-ultra metric on *X*.

There is another beautiful aspect that the whole theory defined earlier remains valid and becomes a part of the new one for example transitive closures will now be defined as the fuzzy binary relation with a value of transitivity 1 which contains the given relation. Moreover for a fuzzy binary relation *R* with transitivity value equal to 1, $\overline{R} = R$ holds.

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An interdisciplinary approach to coalition formation

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This paper concerns an interdisciplinary approach to coalition formation. We apply the MacBeth software (see also [7]), relational algebra, the RelView tool (see [3]), graph theory, bargaining theory, social choice theory (see [4]), and consensus reaching (see also [6]) to the model of coalition formation introduced in [9].

In [9], the notion of a feasible stable government is central. Roughly speaking, a feasible government is a pair consisting of a (majority) coalition of parties and a policy supported by this coalition. Different governments may have different utilities (values) for different parties. Stability of a feasible government means that it is not dominated by another feasible one.

MacBeth, which stands for 'Measuring Attractiveness by a Categorical Based Evaluation Technique', is an interactive approach to quantify the attractiveness of different alternatives, in such a way that the measurement scale constructed is an interval scale; see [1], [2] or www.m-macbeth.com. The MacBeth technique may be applied to many real life situations, and it appears to be a very useful tool also for coalition formation. In [7], we present an application of the MacBeth approach to the model of coalition formation presented in [9]. The MacBeth software increases the applicability of the coalition formation model considerably. We use the MacBeth technique to quantify the attractiveness and repulsiveness of feasible governments to parties. Using MacBeth, every party judges the difference of attractiveness between each two policies on a given issue (including the majority coalitions). The MacBeth software signals when the matrix of judgements of a party becomes inconsistent, and it gives suggestions to make it consistent. We use the MacBeth software to calculate the utilities of governments to parties. Based on these utilities, stable governments are determined. In the original model presented in [9], a party is assumed to express its preferences very precisely. However, the MacBeth tool enables us also to deal with fuzzy preferences.

Since some decades relational algebra is used successfully for formal problem specification, prototyping, and algorithm development. Also, relational algebra seems to be promising for computeraided investigations of coalition formation. In [3], we present an application of relational algebra to coalition formation. We formulate the notions of feasibility, dominance, and stability for governments in relation-algebraic terms. Feasibility of a government can be described by two relations, which state whether a party accepts a coalition and whether a party supports a policy. Stability can be defined in terms of the ‘is-part-of’ relations between parties and governments, the dominance relation on governments, and a list of relations comparing governments with respect to the utility of parties. This enables us to use RelView, a tool for the visualization and manipulation of relations and for prototyping and relational programming, to compute the dominance relation and the set of all feasible stable governments. To illustrate the power of the approach, we solve an example based on the structure of the Polish government after the 2001 elections.

A stable government is by definition not dominated by any other government. However, it may happen that all governments are dominated. In [4], we deal with the problem what to do when there is no un-dominated government. We combine concepts from graph theory, bargaining theory and social choice theory to solve this problem. Using graph-theoretic terms, the non-existence of a stable government means that the dominance graph does not possess a source. Using concepts of graph theory (initial strongly connected components, minimum feedback vertex sets), we present a procedure for choosing one government if the set of all stable governments is empty. As in [3], also in [4] the decisive parts of our procedure are formulated as relational expressions and programs, respectively, so that RelView can be used for executing them and for visualizing the results. Given a dominance graph without a source, first we compute all initial strongly connected components. Next, for each initial strongly connected component, we compute the set of all minimum feedback vertex sets, where a feedback vertex set is a minimal set of vertices which breaks all cycles. Next, we choose a specific minimum feedback vertex set according to the following rule. First, we choose the set(s) for which the number of ingoing arcs is maximal. Since an ingoing arc denotes that a government is dominated, such a choice means selecting governments dominated most frequently. Next, if there are at least two such sets, we choose the one(s) for which the number of outgoing arcs is minimal, meaning the choice of the governments which dominate other governments least frequently. Next, we break all cycles by removing the chosen set of governments. One may say that we remove governments which are least attractive for two reasons: because they are most frequently dominated and they dominate other governments least frequently. According to our procedure, if there is more than one initial strongly connected component, we select the final stable government (from the results of the procedure described above) by applying bargaining or some well-known social choice rules. Concerning the application of bargaining, we use several bargaining games (defined in [8]) and choose the government which is a subgame perfect equilibrium result. Concerning the application of social choice theory, we apply the plurality rule, the majority rule, the Borda rule, or approval voting. Of course, some of these applications may also lead to a non-unique solution. In this case, we propose to combine several techniques and to apply a several-steps method consisting of, for instance, a social choice rule in the first step, and a bargaining game in the second step.

In the model presented in [9], a party evaluates all governments the party belongs to with respect to some criteria. We allow the criteria to be of unequal importance to a party. These criteria concern majority coalitions and policy issues. The parties’ preferences are supposed to be constant, and no possibility of adjusting the preferences of a party is considered. In [6], we introduce a dynamic model of coalition formation in which parties may compromise in order to reach consensus. We apply a consensus model analyzed in [5], where the authors study the problem of formalizing consensus, within a set of decision makers trying to agree on a mutual decision. By combining some notions of both the consensus model [5] and the model of a stable government [9], a new consensus model of political decision-making is constructed. Parties may be advised to adjust their preferences, i.e., to change their evaluation concerning some government(s) or/and the importance of the criteria, in order to obtain a better political consensus. If parties are willing to compromise, it is always possible to reach consensus, and to create a feasible government. In the procedure there is an 'outsider', called the chairman, who advises parties how to adjust their preferences.

First, each feasible coalition tries to reach consensus within this coalition about the government to be formed. Parties consider only feasible governments, i.e., governments acceptable for all parties belonging to the coalition involved, and if there is only one feasible government they can form, they agree. If the parties from a given coalition manage to reach consensus, the coalition proposes to form the government agreed upon. This consensus government is stable in the given coalition with respect to the set of all feasible governments formed by that coalition.

If there are at least two coalitions that succeed in reaching consensus, that is, if at least two governments are proposed, we select the governments which are 'internally stable'. Next, if there are at least two such internally stable governments, some extra procedures are applied in order to choose one of these governments. The protocol given in [6] can be mechanized, resulting in a decision support system for coalition-government formation. The informational requirements of the proposed protocol are demanding, but the MacBeth software can deliver all information needed in a very rational way.

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The Möbius Value: A Generalized Solution for Cooperative Games*

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Abstract. All quasivalues rest on a set of three basic axioms (efficiency, null player, and additivity), which are augmented with positivity for random order values, and with positivity and partnership for weighted values. We introduce the concept of Möbius value associated with a sharing system and show that this value is characterized by the above three axioms. We then establish that (i) a Möbius value is a random order value if and only if the sharing system is stochastically rationalizable and (ii) a Möbius value is a weighted value if and only if the sharing system satisfies the Luce choice axiom.

1 Introduction

The general question raised by any cooperative game can be described as follows: how should the utility sets available to all coalitions be used to determine an outcome from the set of feasible solutions? So far, no single solution-concept has emerged that satisfies everyone's sense of equity (Moulin, 1988). Yet, there seems to be a large agreement to consider the Shapley value as one of the most appealing solutions (Shapley, 1953). However, when players do not stand behind the veil of ignorance, this solution is no longer valid. Various concepts have then been proposed to deal with social and economic contexts in which players have idiosyncratic rights in sharing the final outcome (see Monderer and Samet, 2002, for a recent survey). All these solutions rest on a common set of three basic axioms (efficiency, null player, and additivity), which are augmented with positivity by Weber (1988) in the case of random order values, and with positivity and partnership by Kalai and Samet (1987) for weighted values. In this paper, we restrict ourselves to these three basic axioms only and characterize the set of corresponding values that we call *Möbius values*.

The extensions of the Shapley value allow for a redistribution of the total worth according to two dimensions: the marginal contribution of each player within all possible coalitions and a *sharing system* which is given a priori. The idea behind the sharing system is that the reward of a player may be related to her marginal contribution to each coalition in various ways. This aims at capturing the fact that a society may be governed according to a large variety of distributive rules, which are themselves based on various principles of justice (Bentham, Rawls, etc.). For example, in the theory of cooperative values, Kalai and Samet (1987) attribute a given weight to each player that expresses her power within each coalition whereas, in Weber (1988), the weight depends on the relative place of the player in society endowed with different orderings. As will be shown in this paper, the additional axioms that have been introduced in the literature (positivity and partnership) do actually restrict in a fairly strong manner the admissible sharing systems. More precisely, we will see that *the existing values*, called quasivalues, *are such that the sharing rule within a particular coalition is constrained by the way the sharing rule is defined within all broader coalitions (and vice versa).* Put differently, saying how to share within the grand coalition tells us how to share within all subcoalitions. In practice, the

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existence of such a master sharing rule may be problematic because it requires the implicit agreement of all players about it. By contrast, *our approach allows for sharing systems that are independent of the existence of such a master sharing rule and is, therefore, more general*. Yet, we need some minimal requirement linking sharing in a coalition and sharing in its subcoalitions. In this respect, we suggest that the way the worth of a coalition is shared is such that, in all its subcoalitions, players' shares cannot be lower than what they are in the referential coalition. Given this caveat, sharing within a particular coalition need not be related to the way the worth of any subcoalition or supercoalition is distributed among its members. In other words, by being the members of a coalition, the corresponding players find themselves in a particular sharing context that defines what each of them will receive. It is in that sense that the Möbius value allows for sharing when context matters.

In our paper, the Möbius value of a player is given by a linear combination of the pure contribution of her cooperation within all coalitions including her; the coefficient associated with each coalition is the share that this player can claim in this coalition. By "pure contribution of cooperation" (**PCC**), we mean the net reward of cooperation within a coalition after having discounted for what cooperation brings about in all possible proper subcoalitions. Formally, the **PCC** of a coalition is the Möbius inverse of the characteristic function of the game. Focussing on the **PCC** of a coalition, instead of the marginal contribution of its members, concurs with our idea that a coalition defines a specific sharing context, which is a priori independent of all possible subcoalitions. Moreover, the coefficients of the linear combination define a probability over the corresponding coalition, but they need not be "consistent" across coalitions. By contrast, we identify two forms of consistency of the sharing system used in quasivalues. First, *a random order value is such that the sharing system is stochastically rationalizable*, that is, there exists a probability distribution defined over all orderings on the set of players which yields the sharing system (Block and Marschack, 1960). Second, *a weighted value*, as introduced by Kalai and Samet (1987), *is such that the sharing system satisfies the more demanding condition given by the Luce choice axiom used in discrete choice theory* (Luce, 1959). This axiom says that each sharing rule may be viewed as the Bayesian restriction of a master distribution defined on the set of players. Hence, our approach to cooperative values allows us to characterize each quasivalue by means of restrictions imposed on the corresponding sharing system.[3]

The remainder of this paper is organized as follows. Definitions and notation are given in Section 2. The concept of a Möbius value is defined and axiomatically characterized in Section 3 (Theorem 1). The relationships with quasivalues are explored in Section 4 where the following results are proven: (i) a Möbius value is a random order value if and only if the sharing system is stochastically rationalizable (Theorem 3) and (ii) a Möbius value is a weighted value if and only if the sharing system satisfies the Luce choice axiom (Theorem 4). In Section 4, we prove that a Möbius value is positive if and only if the game is monotone (Theorem 5) and that the set of Möbius values is the core if and only if the game is convex (Theorem 7). Section 6 concludes.

2 The Pure Contribution of Cooperation in a TU-Game

A *cooperative game with transferable utility* (TU-game) is a pair (Z,ν) where Z, the *grand coalition* with $\sharp Z = n$, is defined by a finite set of *players* and ν, the *characteristic function*, is defined by a mapping from 2^Z to $\mathbb{R}$ such that $\nu(\emptyset) = 0$. Any subset Y of Z is called a *coalition* and for any nonempty coalition Y, we denote $Z \backslash Y$ by $\overline{Y}$, $Y \backslash \{i\}$ by Y_{-i}, $Y \cup \{i\}$ by Y_{+i} and $2^Y \backslash \emptyset$ by $2^Y_{-\emptyset}$.

[3] These results also uncover some new connections between cooperative values and probabilistic discrete choices, a topic which has already been under investigation (Monderer, 1992; Gilboa and Monderer, 1992).

The set of TU-games whose set of players is Z is given by the vector space $\mathbb{R}^{2^{Z}_{-\emptyset}}$. A characteristic function ν is *monotone* if $\nu(X) \leq \nu(Y)$ for every $X \subset Y$ and *convex* if $\nu(X \cup Y) + \nu(X \cap Y) \geq \nu(X) + \nu(Y)$ for every pair $X, Y \in Z$. For convenience, all properties that are satisfied by ν on Z are said to be satisfied by the TU-game itself.

A *solution* of the game (Z, ν) is a mapping $\varphi : \mathbb{R}^{2^{Z}_{-\emptyset}} \to \mathbb{R}^n$. A solution $\varphi(\nu)$ is said to be *positive* when $\varphi_i(\nu) \geq 0$ for all $i \in Z$.

The above concepts are standard and we now introduce one of the new tools of this paper.

Consider any TU-game (Z, ν). Then, for any nonempty coalition Y, following Shapley (1953), there exists a unique set of coefficients $\left(\Gamma_\nu(X) : X \in 2^{Y}_{-\emptyset}\right)$ such that:

$$\nu(Y) = \sum_{X \in 2^{Y}_{-\emptyset}} \Gamma_\nu(X) \tag{1}$$

that are given by

$$\Gamma_\nu(Y) = \sum_{X \in 2^{Y}_{-\emptyset}} (-1)^{y-x} \nu(X) \tag{2}$$

where y and x stand for the cardinalities of Y and X, respectively. These coefficients may be interpreted as follows. Set $\nu(i) \equiv \nu(\{i\})$. If $Y = \{i, j\} \subset Z$, the worth $\nu(Y)$ may be different from $[\nu(i) + \nu(j)]$. In such a context, two cases may arise. In the first, the cooperation is "negative" because the two players are worse off when they cooperate. In the second, the cooperation is "positive" because the two players are better off when they cooperate. In both cases, it is natural to express the *pure contribution of cooperation* **(PCC)** $\Gamma_\nu(Y)$ *of* Y, also called the *dividend* of Y in Harsanyi (1963), by the difference

$$\Gamma_\nu(Y) = \nu(Y) - [\nu(i) + \nu(j)]. \tag{3}$$

In other words, $\Gamma_\nu(Y)$ measures the exact contribution of the cooperation inside of Y because we have accounted for the individual worthies. When $Y = \{i, j, k\}$, one might think that $\Gamma_\nu(Y)$ would be given by $\Gamma_\nu(Y) = \nu(Y) - [\nu(i) + \nu(j) + \nu(k)]$. However, this expression already includes the **PCC** of each pair $\{i, j\}$, $\{i, k\}$ and $\{j, k\}$ to the **PCC** of Y. Given (3), the **PCC** of $Y = \{i, j, k\}$ should instead be defined as follows:

$$\begin{aligned} \Gamma_\nu(Y) = &\{\nu(Y) - [\nu(i) + \nu(j) + \nu(k)]\} \\ &- \{\Gamma_\nu(i, j) + \Gamma_\nu(i, k) + \Gamma_\nu(j, k)\}. \end{aligned} \tag{4}$$

More generally, in view of these expressions, we define the **PCC** of a TU-game as the mapping $\Gamma_\nu : 2^{Z}_{-\emptyset} \to \mathbb{R}$ such that, for each coalition $Y \subset Z$, (1) and (2) hold. In words, $\Gamma_\nu(Y)$ can be interpreted as *the contribution of cooperation within the coalition Y independently of what cooperation brings about in all possible subcoalitions that could have been formed before the coalition Y is determined*. Stated differently, $\Gamma_\nu(Y)$ measures the total benefit generated by the coalition Y once we have accounted for all the possible subcoalitions formed by any proper subset of players.[4]

The **PCC** Γ_ν is equivalent to the Möbius inverse of the characteristic function ν (Rota, 1964; Chateauneuf and Jaffray, 1992). Note that, for any $Y \in 2^{Z}_{-\emptyset}$, we have:

$$\nu(Y) = \sum_{i \in Y} \nu(i) + \sum_{\substack{X \subset Y \\ x \geq 2}} \Gamma_\nu(X) \tag{5}$$

[4] The **PCC** of a coalition is the game-theoretic counterpart of the "contextual utility" as defined by Billot and Thisse (1999) in discrete choice theory and of the "evidence of an event" in Dempster-Shafer's theory of belief functions.

which means that the worth of a coalition is equal to the sum of the individual worthies plus the sum of the PCCs of all possible subcoalitions. In particular, for the grand coalition, we have:

$$\nu(Z) = \sum_{Y \subset Z} \Gamma_\nu(Y)$$

that is, the worth of the grand coalition is equal to the sum of the pure contributions of all possible coalitions.

In what follows, we show that the **PCC** of a coalition may be negative even when the TU-game is monotone. The same example is used throughout the paper.

Example 1: Consider the TU-game (Z, ν) such that $Z = \{1,2,3\}$ whereas its characteristic function v is defined by

$$\begin{cases} \nu(Z) = 8, \\ \nu(Z_{-i}) = 7 - i, \qquad \forall i \in Z \\ \nu(i) = i, \qquad \forall i \in Z. \end{cases}$$

This characteristic function is monotone and convex. The associated **PCCs** can be computed as follows:

$$\begin{cases} \Gamma_\nu(Z) = \Gamma_\nu(123) = 8 - (6+5+4) + (1+2+3) = -1, \\ \Gamma_\nu(Z_{-1}) = \Gamma_\nu(23) = 6 - (2+3) = 1, \\ \Gamma_\nu(Z_{-2}) = \Gamma_\nu(13) = 5 - (1+3) = 1, \\ \Gamma_\nu(Z_{-3}) = \Gamma_\nu(12) = 4 - (1+2) = 1, \\ \Gamma_\nu(1) = 1, \\ \Gamma_\nu(2) = 2, \\ \Gamma_\nu(3) = 3. \end{cases}$$

This implies that (i) the **PCC** of a pair Z_{-i} is greater than that of the grand coalition Z, (ii) the **PCC** of a pair is constant whoever is in the pair, and (iii) the **PCC** of the grand coalition is negative. Note also that

$$\begin{aligned} \nu(Z) &= \sum_{Y \subset Z} \Gamma_\nu(Y) \\ &= -1 + (1+1+1) + (1+2+3) \\ &= 8 \end{aligned}$$

while

$$\begin{cases} \nu(Z_{-1}) = \nu(23) = 1+2+3 = 6, \\ \nu(Z_{-2}) = \nu(13) = 1+1+3 = 5, \\ \nu(Z_{-3}) = \nu(12) = 1+1+2 = 4. \end{cases}$$

3 Möbius Values

3.1 Definition

The *sharing rule* of a coalition $Y \in 2^Z_{-\emptyset}$ is a probability distribution $p_Y : 2^Y \to [0,1]$ where $p_Y(i)$ corresponds to the share player $i \in Y \in 2^Z_{-\emptyset}$ may claim in coalition Y, which satisfies the following two conditions:

Individual sharing consistency : For all coalitions $X \subset Y \in 2^Z_{-\emptyset}$ and all player $i \in X$, $p_Y(i) \leq p_X(i)$.

Negligible player condition : If $p_{\{i,j\}}(i) = 0$ for some $i, j \in Y$, then for all coalitions $X \subset Y \in 2^Z_{-\emptyset}$, $p_Y(X) = p_{Y_{-i}}(X_{-i})$.

The first condition implies that, when a coalition shrinks, an individual's shares never decreases. This precludes sharing context in which the departure of one individual from a group would reduce the share the others can claim. The second condition means that a player gets a zero share in any coalition when she gets such a share in a 2-person coalition. The latter condition implies that $p_Y(i) = 0$ for each negligible player, whereas $p_{\{i\}}(i) = 1$ for each player $i \in Z$.

A *sharing system*, denoted $(Z, \mathcal{P})$, is then defined by the set Z of players and by a mapping which associates each coalition $Y \in 2^Z_{-\emptyset}$ with a sharing rule p_Y.

Apart from the two conditions above, the sharing rule p_Y depends only upon the particular redistribution context defined by the coalition. Hence, for any coalition X different from Y, p_X need not be functionally related to p_Y in the system $(Z, \mathcal{P})$. As will be seen below, this makes our approach to cooperative values more general than standard quasivalues (Monderer and Samet, 2002).

Consider a TU-game (Z, ν). We define the *Möbius value* of the player $i \in Z$ associated with the sharing system $(Z, \mathcal{P})$, denoted $\varphi_i(\nu, \mathcal{P})$ by

$$\varphi_i(\nu, \mathcal{P}) = \sum_{\substack{Y \in 2^Z_{-\emptyset} \\ Y \ni i}} p_Y(i) \Gamma_\nu(Y). \tag{6}$$

In words, the Möbius value of player i is given by a linear combination of the **PCCs** of all nonempty coalitions Y including i, where the coefficient $p_Y(i)$ associated with the coalition Y is the share that player i can claim in this coalition.[5] When the expression above holds for all sharing systems, we discard $\mathcal{P}$ in $\varphi_i(\nu, \mathcal{P})$; similarly, we denote p_Z by p.

Remark 1: For any negligible player i, we have $\varphi_i(\nu) = \nu(i)$.

Example 2: Consider a sharing rule $\mathcal{P}^*$ given by $p^*(1) = p^*(2) = p^*(3) = 1/3$, $p^*_{12}(1) = 1/3$ while $p^*_{12}(2) = 2/3$, $p^*_{13}(1) = 2/3$ while $p^*_{13}(3) = 1/3$ and $p^*_{23}(2) = 2/3$ while $p^*_{23}(3) = 1/3$. Then, the associated Möbius value $\varphi_i(\nu, \mathcal{P}^*)$ defined by (17) leads to

$$\begin{aligned}
\varphi_1(\nu, \mathcal{P}^*) &= \Gamma_\nu(1)\, p^*_1(1) + \Gamma_\nu(12)\, p^*_{12}(1) + \Gamma_\nu(13)\, p^*_{13}(1) \\
&\quad + \Gamma_\nu(Z)\, p^*(1) \\
&= 1 + \frac{1}{3} + \frac{2}{3} - \frac{1}{3} = \frac{5}{3} \simeq 1.65,
\end{aligned}$$

$$\begin{aligned}
\varphi_2(\nu, \mathcal{P}^*) &= \Gamma_\nu(2)\, p^*_2(2) + \Gamma_\nu(12)\, p^*_{12}(2) + \Gamma_\nu(23)\, p^*_{23}(2) \\
&\quad + \Gamma_\nu(Z)\, p^*(2) \\
&= 2 + \frac{2}{3} + \frac{2}{3} - \frac{1}{3} = 3,
\end{aligned}$$

and

$$\begin{aligned}
\varphi_3(\nu, \mathcal{P}^*) &= \Gamma_\nu(3)\, p^*_3(3) + \Gamma_\nu(13)\, p^*_{13}(3) + \Gamma_\nu(23)\, p^*_{23}(3) \\
&\quad + \Gamma_\nu(Z)\, p^*(3) \\
&= 3 + \frac{1}{3} + \frac{1}{3} - \frac{1}{3} = \frac{10}{3} \simeq 3.35.
\end{aligned}$$

[5] A related, but different, approach in terms of interaction among players forming a coalition is developed by Grabisch and Roubens (1999).

The Möbius solution of our example is therefore given by the triplet

$$(1.65, 3, 3.35).$$

Note that, in this example, p^*_{12}, p^*_{13} and p^*_{23} are *not* Bayesian restrictions of p^* onto the subsets $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$. In other words, the sharing rule p^*_{12}, p^*_{13} and p^*_{23} are independent of the master rule p^*.

3.2 Axioms

We introduce the following three axioms to characterize the Möbius value as defined by (6).

Axiom 1 (φ-Efficiency) : *Let (Z,ν) be any TU-game. Then,*

$$\varphi_Z(\nu) = \sum_{i\in Z} \varphi_i(\nu) = \nu(Z).$$

This means that the solution of the grand coalition is equal to its worth.

Axiom 2 (φ-Null Player) : *For each player $i \in Z$, if for each coalition $Y \subset Z_{-i}$ we have $\Gamma_\nu(Y_{+i}) = 0$, then*

$$\varphi_i(\nu) = 0.$$

This axiom says that the solution of an individual is zero when her **PCC** of any coalition she belongs to is always zero. Note that $\nu(i) = 0$ and $\nu(Y_{+i}) = \nu(Y)$ when i is a null player.[6]

Axiom 3 (φ-Linearity) : *Let (Z,ν) and (Z,μ) any two TU-games and $\alpha \in \mathbb{R}$. Then,*

$$\varphi(\alpha\nu + \mu) = \alpha\varphi(\nu) + \varphi(\mu).$$

For any $X \subset Z$, consider a X*-unanimity TU-game* (Z,ν^X) for which the characteristic function ν^X is defined as follows:

$$\nu^X(Y) = \begin{cases} 1 & \text{if } X \subset Y, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Lemma 1. : *For any coalition $X \subset Z$, the **PCC** Γ_{ν^X} associated with the unanimity TU-game (Z,ν^X) is such that:*

$$\Gamma_{\nu^X}(Y) = \begin{cases} 1 & \textit{if } X = Y, \\ 0 & \textit{otherwise.} \end{cases}$$

Lemma 2. : *For any TU-game (Z,ν), we have:*

$$\nu = \sum_{X\in 2^Z_{-\emptyset}} \Gamma_\nu(X)\nu^X.$$

Proofs are straightforward and omitted. Note that Shapley (1953, p. 311) already proved that every characteristic function can be decomposed in a unique way as a linear combination of unanimity games (our Lemma 2).

[6] This axiom is weaker than the dummy axiom used by Shapley (1953) and Weber (1988). See Nowak and Radzik (1994) and Monderer and Samet (2002) for a discussion of the null player axiom *vs* the dummy axiom.

Lemma 3. : *Let i be any player of Z. Under* A1-A3, *for any nonempty Y, we have:*

$$\varphi_i\left(\alpha\nu^Y\right) = \begin{cases} \alpha p_Y(i) & \text{if } i \in Y, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: By A3, we may assume without loss of generality that $\alpha = 1$. First, consider a given coalition Y. If $i \notin Y$, then, for each subcoalition X such that $i \notin X$, (7) implies that $\nu^Y(X_{+i}) = \nu^Y(X) = 0$. Thus, any player $i \in Z \backslash Y$ is a null player for ν^Y. Hence, by A2, we have $\varphi_j\left(\nu^Y\right) = 0$ for all $i \notin Y$.

Second, if $i \in Y$, it follows from (6) that

$$\varphi_i\left(\nu^Y\right) = \sum_{\substack{X \subset Y \\ X \ni i}} p_X(i)\Gamma_{\nu^Y}(X).$$

Now, Lemma 1 implies that $\Gamma_{\nu^Y}(X) = 0$ for all $X \neq Y$ and $\Gamma_{\nu^Y}(X) = 1$ for $X = Y$. Consequently, we obtain for all $i \in Y$:

$$\varphi_i\left(\nu^Y\right) = p_Y(i)\Gamma_{\nu^Y}(Y) = p_Y(i).$$

□

This result also shows the existence of a one-to-one correspondence between the Möbius values and the sharing systems.

We may now state one of our main results.

Theorem 1. : *Any solution* $\varphi(\nu)$ *of the TU-game* (Z,ν) *is a Möbius value if and only if* $\varphi(\nu)$ *satisfies the axioms* A1-A3.

Proof: (Sufficiency) Using Lemma 2 and A3, we have:

$$\varphi_i(\nu) = \sum_{Y \in 2^Z_{-\emptyset}} \varphi_i\left[\Gamma_\nu(Y)\nu^Y\right].$$

Hence, from Lemma 3 and A3, it follows that

$$\begin{aligned}\varphi_i(\nu) &= \sum_{Y \subset Z_{-i}} \sum_{X \subset Y} \varphi_i\left[\Gamma_\nu(X_{+i})\nu^{X_{+i}}\right] \\ &= \sum_{X \ni i} \Gamma_\nu(X)\, p_X(i)\end{aligned}$$

which is identical to (6).

(Necessity) The proof is straightforward.

□

3.3 The Shapley Value as a Uniform Möbius Value

The Shapley value of a TU-game (Z,ν), denoted $S(\nu)$, allocates the worth $\nu(Z)$ among all players $i \in Z$ as follows:

$$S_i(\nu) = \frac{1}{n!} \sum_{\substack{X \subset Z \\ i \in X}} (x-1)!\,(n-x)![\nu(X) - \nu(X_{-i})]. \tag{8}$$

The standard interpretation of the Shapley value is as follows. Assume that the players in Z are randomly ordered as $(i_1, i_2, \cdots, i_n)$ such that each ordering is equally probable. The Shapley value $S_i(\nu)$ is then the average of player i's marginal contributions $\nu(X) - \nu(X_{-i})$ taken over all coalitions $X \subset Z$.

The probability of any coalition X is defined by the probability that the predecessors of i in the random ordering $(i_1, i_2, \cdots, i_n)$ are the elements of X.

Our next result suggests another interpretation: when player i cooperates within a coalition X whose **PCC** equals $\Gamma_\nu(X)$, player i gets the same "share" from this coalition than any other member of X. In other words, the sharing of $\Gamma_\nu(X)$ is *uniform* within X. Hence, the Shapley value of player i is the unweighted and normalized sum of all coalition worthies. The associated sharing system is denoted $(Z, \mathcal{U})$ where $\mathcal{U} = \left(u_Y : Y \in 2^Z_{-\emptyset}\right)$ and u_Y the uniform probability distribution over Y.

As shown by Denneberg and Grabisch (1999, Theorem 4.1), this result can be proven by using symmetry, whereas Grabisch (1997) gives a direct proof in a setting involving interactions among individuals. Yet, in order to illustrate the nature of individual contributions to a coalition, we give in Appendix B a different proof that does not rely on symmetry.

Theorem 2. : *Let $(Z, \mathcal{U})$ be the uniform sharing system. Then, the corresponding Möbius value of the TU-game (Z, ν) is the Shapley value:*

$$\varphi_i(\nu, \mathcal{U}) = \sum_{\substack{Y \in 2^Z_{-\emptyset} \\ Y \ni i}} \frac{\Gamma_\nu(Y)}{y} = S_i(\nu) \quad \textit{for all players } i \in Z.$$

In other words, the Shapley value corresponds to a sharing of the **PCC**s which is uniform across players. This interpretation is perfectly consistent with the axiom of anonymity (or symmetry) which defines the Shapley value (Shapley, 1953): players are a priori given the same share in *all* possible coalitions. This should not come as a surprise since, on the one hand, we know from Kalai and Samet (1987) that the Shapley value is a weighted value with identical weights and, on the other hand, that the uniform distribution satisfies the Luce choice axiom that characterizes weighted values (see our Theorem 4 below).

Example 3: Consider the uniform sharing rule given by $u_X(i) = 1/x$ for all $i \in X$, all $X \subset Z$. Then, the Shapley value $S_i(\nu)$ defined by (17) leads to

$$\begin{aligned} S_1(\nu) &= \Gamma_\nu(1)\, u_1(1) + \Gamma_\nu(12)\, u_{12}(1) + \Gamma_\nu(13)\, u_{13}(1) \\ &\quad + \Gamma_\nu(Z)\, u(1) \\ &= 1 \times 1 + 1 \times \frac{1}{2} + 1 \times \frac{1}{2} - 1 \times \frac{1}{3} \\ &= 1 + \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = \frac{5}{3} \simeq 1.66 > 1, \end{aligned}$$

$$\begin{aligned} S_2(\nu) &= \Gamma_\nu(2)\, u_2(2) + \Gamma_\nu(12)\, u_{12}(2) + \Gamma_\nu(23)\, u_{23}(2) \\ &\quad + \Gamma_\nu(Z)\, u(2) \\ &= 2 + \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = \frac{8}{3} \simeq 2.66 > 2 \end{aligned}$$

and

$$\begin{aligned} S_3(\nu) &= \Gamma_\nu(3)\, u_3(3) + \Gamma_\nu(13)\, u_{13}(3) + \Gamma_\nu(23)\, u_{23}(3) \\ &\quad + \Gamma_\nu(Z)\, u(3) \\ &= 3 + \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = \frac{11}{3} \simeq 3.66 > 3. \end{aligned}$$

The Shapley solution of our example is therefore given by the triplet

$$(1.66, 2.66, 3.66).$$

4 Relationships between Möbius Values and Quasivalues

4.1 Random Order Values

Weber (1988) has introduced a generalization of the Shapley value, called *random order (or probabilistic) values*, by weighting the marginal contributions $\nu(Y_{+i}) - \nu(Y)$ of player i by the probability π_Y^i of joining any coalition Y in Z_{-i}:

$$\phi_i(\nu) = \sum_{Y \subset Z_{-i}} \pi_Y^i [\nu(Y_{+i}) - \nu(Y)]. \tag{9}$$

Then, Weber (1988) has proved that a solution is a quasivalue if and only if it is a random order value.

When comparing (6) and (9), we first note that the coefficients π_Y^i in (9) are interpreted by Weber as the probability for i to become a member of Y (or to join Y) while, in the present paper, $p_Y(i)$ in (6) is defined as the share attributable to player i when i is a member of the coalition Y. The two interpretations are therefore different. Second, the marginal contribution $\nu(Y_{+i}) - \nu(Y)$ differs from the pure contribution $\Gamma_\nu(Y_{+i})$ of coalition Y_{+i}. So, the connection between the two values is not clear (at least to us). Hence, our research strategy is naturally to uncover the relationships between (6) and (9) through their respective coefficients. More precisely, we are interested in determining *the connections between the share a player may obtain within a particular coalition and the probability she has to join this coalition*.

Definition (6) may be rewritten in terms of marginal contribution as follows:

$$\begin{aligned}
\varphi_i(\nu) &= \sum_{Y \in 2^Z_{-\emptyset}} \Gamma_\nu(Y)\, p_Y(i) \\
&= \sum_{\substack{Y \subset Z \\ i \in Y}} p_Y(i) \sum_{X \subset Y} (-1)^{y-x} \nu(X) \\
&= \sum_{X \subset Z} \nu(X) \sum_{Y \supset X_{+i}} (-1)^{y-x} p_Y(i)
\end{aligned}$$

that is

$$\varphi_i(\nu) = \sum_{Y \subset Z_{-i}} \left\{ \sum_{X \supset Y_{+i}} (-1)^{x-(y+1)} p_X(i) \right\} [\nu(Y_{+i}) - \nu(Y)]. \tag{10}$$

This shows that the Möbius value involves coefficients γ_Y^i of the marginal contributions of i to Y of the type

$$\gamma_Y^i \equiv \sum_{X \supset Y_{+i}} (-1)^{x-y-1} p_X(i), \text{ for all } Y \subset Z_{-i} \tag{11}$$

which are not here primitives of the game, as they are in the various extensions of the Shapley value (Monderer and Samet, 2002). Furthermore, γ_Y^i need not be a probability and may even be negative. As will been seen, all quasivalues are special cases of the Möbius value in which the coefficients γ_Y^i take a particular form. Stated differently, *all quasivalues are special Möbius values associated with specific sharing systems*. In particular, the Möbius value is a random order value if and only if the coefficients γ_Y^i are probabilities. In this case, whenever the game is monotone, the positivity axiom for quasivalues - which one can find from Kalai and Samet (1987) to Monderer and Samet (2002) through Weber (1988) - always holds. Hence, it remains to identify the restrictions to be imposed on the sharing system for a Möbius value to have probabilistic coefficients.

If $Y = Z_{-i}$, then the coefficient π^i_Y for player i to join the coalition Y is identical to her share $p(i)$. Consider now $Y = Z_{-ij}$. Once i has joined Y, either i belongs to the coalition Z_{-j} or to the coalition Z because Y_{+i} is a subset of both. Since $Y = Z_{-ij}$, the weight for i to join Y is therefore given by:

$$\pi^i_Y = p_{Z_{-j}}(i) - p(i). \tag{12}$$

In other words, π^i_Y is the coefficient of joining the coalition Y without being in the coalition Z. If $Y = Z_{-ijk}$, one might think that π^i_Y is such that

$$\pi^i_Y = p_{Z_{-jk}}(i) - p_{Z_{-j}}(i) - p_{Z_{-k}}(i) - p(i).$$

However, this expression does not account for the fact that, when i belongs to Z_{-j} (resp. Z_{-k}), this may be because she has joined Z_{-ij} (resp. Z_{-ik}). Deleting these occurrences, we obtain:

$$\pi^i_Y = p_{Z_{-jk}}(i) - \left[p_{Z_{-j}}(i) - \pi^i_{Z_{-ij}}\right] - \left[p_{Z_{-k}}(i) - \pi^i_{Z_{-ik}}\right] - p(i).$$

Given (12), this may be rewritten as follows:

$$\pi^i_Y = p_{Z_{-jk}}(i) - p_{Z_{-j}}(i) - p_{Z_{-k}}(i) + p(i).$$

More generally, for all $i \in Z$ and all $Y, X \subset Z_{-i}$, the coefficient for i to join Y is given by:

$$\pi^i_Y = \sum_{X \supset Y} (-1)^{x-y} p_{X_{+i}}(i).$$

It is readily verified that this expression can be also written as follows:

$$\pi^i_Y = p_{Y_{+i}}(i) - \sum_{X \supsetneq Y} \left[p_{X_{+i}}(i) - \pi^i_X\right]$$

where $\pi^i_Z \equiv 0$.

The difference $p_{X_{+i}}(i) - \pi^i_X$ may be viewed as the *net share* of player i for being in X_{+i}, once π^i_X is interpreted as the (normalized) "cost" she bears to join the coalition X. Then, the coefficient for i to join the coalition Y is equal to her share in the coalition Y_{+i} minus the sum of the net shares that i belongs to all the supercoalitions $X_{+i} \supset Y$. Put differently, π^i_Y *is the coefficient to join Y directly and not through any of its supercoalitions*. Using again (1) and (2), we then have: for all $i \in Z$, all $Y, X \subset Z_{-i}$,

$$\pi^i_Y = \sum_{X \supset Y} (-1)^{x-y} p_{X_{+i}}(i) \tag{13}$$

if and only if

$$p_{Y_{+i}}(i) = \sum_{X \supset Y} \pi^i_X. \tag{14}$$

Remark 2: Expressions (13) and (14) can be viewed as, respectively, the Möbius and the *co-Möbius inverse* of a set function v^i (which is unrelated to the characteristic function ν), such that the following two conditions hold (see Appendix A for more details):

$$\begin{cases} p_{Y_{+i}}(i) = \sum_{X \supset \overline{Y}} (-1)^{z-x} v^i(X), \\ v^i(Y) = \sum_{X \supset \overline{Y}} (-1)^x p_{X_{+i}}(i). \end{cases} \tag{15}$$

Hence, the properties of the Möbius and co-Möbius inverse of v^i can be used for studying the relationships between π_Y^i and $p_{Y_{+i}}(i)$ where the former corresponds to the Möbius inverse and the latter to the co-Möbius (see, e.g. the proof of Theorem 5).

Remark 3: Expression (14) may be given the following interpretation: the share of player i in Y_{+i} is equal to the sum of the coefficients that this player has to join all supercoalitions of Y, that is, *her share must cover exactly the sum of the costs that she would incur by joining all the supercoalitions of Y*.

Remark 4: In the special case where there exist some players i such that $p(i) = 0$, then (13) implies $\pi_Y^i = 0$ for all coalitions $Y \neq \emptyset$. In other words, all such players always stay alone because $\pi_\emptyset^i = 1$.

Equations (13) and (11) imply

$$\gamma_Y^i = \pi_Y^i.$$

However, for π_Y^i to be a probability, the sharing system $(Z, \mathcal{P})$ must satisfy some additional conditions that we now investigate. Following Block and Marschak (1960) and Falmagne (1978), we say that the sharing system $(Z, \mathcal{P})$ is *stochastically rationalizable* if and only if the Block-Marschak polynomials of $(Z, \mathcal{P})$ are all nonnegative. Recall that the Block-Marschak polynomials of $(Z, \mathcal{P})$ are defined for all subsets $Y \subset Z_{-i}$ by the expression:

$$K(i,Y) = \sum_{k=0}^{y} (-1)^k \sum_{X \in \mathcal{F}(Y, y-k)} p_{\overline{X}}(i)$$

where $\mathcal{F}(Y, y-k)$ is the family of subsets of Y whose cardinal is equal to $y-k$ and $\overline{X}$ the complement of X in Z. We thus have:

Theorem 3. : *For any TU-game (Z, ν), the Möbius value is a random order value, i.e.*

$$\varphi_i(\nu) = \phi_i(\nu),$$

if and only if the sharing system $(Z, \mathcal{P})$ is stochastically rationalizable.

Proof: Expressions (13) and (14) define a one-to-one correspondence between the two sets of coefficients γ_Y^i and π_Y^i. To prove that the coefficients π_Y^i correspond to Weber's probabilities, it remains to show, on one hand, that they are all nonnegative and, on the other hand, that $\sum_{Y \subset Z_{-i}} \pi_Y^i = 1$.

Let X and Y be any two subsets of Z such that $i \notin X$ and $i \in Y$. We have

$$\begin{aligned} K(i,Y) &= \sum_{k=0}^{y} (-1)^k \sum_{X \in \mathcal{F}(Y, y-k)} p_{\overline{X}}(i) \\ &= \sum_{X \subset Y} (-1)^{y-x} p_{\overline{X}}(i) \\ &= \sum_{\overline{X} \supset \overline{Y}} (-1)^{\bar{x}-\bar{y}} p_{\overline{X}}(i) \\ &= \pi_{\overline{Y}}^i. \qquad \text{by (13)} \end{aligned}$$

Since Y is arbitrary, π_Y^i is nonnegative if and only if the sharing system (Z, p) is stochastically rationalizable. Moreover, it is readily verified that $\sum_{Y \subset Z_{-i}} \pi_Y^i = p_i(i) = 1$, which ends the proof.

□

Corollary 1. : *Any Block-Marschak polynomial $K(i,Y)$ of a choice probability system $(Z,\mathcal{P})$ corresponds to the coefficient π_Y^i as defined by (13).*

Theorem 3 is consistent with the following result derived by Monderer (1992): for any random order value, there exists a rationalizable system of choice probabilities defined on Z consistent with the probabilities π_Y^i in (9). Note also that the stochastic rationality of the sharing system $(Z,\mathcal{P})$ is equivalent to the positivity axiom. Then, *a solution satisfying A1-A3 whose sharing system is stochastically rationalizable is a quasivalue*.

Observe that (13) allows for the computation of the coefficients used by Weber from the individual shares. This, in turn, permits the study of the likelihood of various coalitions and, therefore, to analyze the occurrence of coalition formation and to perform some "comparative statics" on the sharing rule. Everything else equal, *the smaller* (resp. the larger) *a player's share, the higher* (resp. the lower) *her probability to stand alone*, a situation which involves no coalitional cost. Likewise, the smaller (resp. the larger) a player's share, the higher (resp. the lower) her probability to be joined by players with larger shares. Unfortunately, it seems hard to say something about players with intermediate shares without specifying the connections between the sharing system $\mathcal{P}$ and the characteristic function ν.

4.2 Weighted Values

Kalai and Samet (1987) have considered a subset of quasivalues defined as follows. Set a *weight system* $w = (w_X)_{X\in 2^Z_{-\emptyset}}$ such as

$$w_X(i) = \frac{w_Y(i)}{w_Y(X)}$$

for all $Y \supset X$, all $i \in X$ and $w_Y(X) > 0$. It is worth noting that a weight system w is *strictly* positive. The associated *weighted value* ϕ^w is then defined for any unanimity game ν^X by

$$\phi_i^w(\nu^X) = \begin{cases} w_X(i) & \text{if } i \in X, \\ 0 & \text{otherwise.} \end{cases}$$

In words, a player belonging to coalition X receives her weight within this coalition. Moreover, a coalition Y is said to be a coalition of partners or a *p-type coalition* in (Z,ν) if, for every subcoalition $X \subset Y$ and each $W \subset \overline{Y}$, $\nu(W \cup X) = \nu(W)$. In other words, players are called partners when they refuse to cooperate outside the coalition of partners. A value ϕ satisfies the *partnership axiom* if, whenever Y is a p-type coalition:

$$\phi_i(\nu) = \phi_i(\phi_Y(\nu)\nu^Y) \qquad \text{for all } i \in Y \tag{16}$$

where ϕ_Y is the share attributed to the coalition Y. This axiom, introduced by Kalai and Samet (1987), requires that if subcoalitions of Y are irrelevant, then it makes no difference either players of Y receive their individual shares in ν, or they altogether receive their group share in ν and determine their individual shares later. Kalai and Samet (1987) then proves that a weighted value is a quasivalue that satisfies the partnership axiom. Hence, we need to identify the properties of the sharing rules which characterize a Möbius value as a weighted value, i.e. to interpret the partnership axiom in terms of shares.

Lemma 4. *: For any TU-game (Z,ν), a Möbius value satisfies the partnership axiom if and only if the sharing system $(Z,\mathcal{P})$ satisfies the Luce choice axiom: for all $Y \in 2^Z_{-\emptyset}$*

$$p(i) = p(Y) \times p_Y(i) \qquad \textit{for all } i \in Z \textit{ such that } 0 < p(i) < 1.$$

Proof: As noticed by Chun (1991, p.186), it is always possible to define the weight system w by $w_i = \varphi_i\left(\nu^Z\right)$ where ν^Z is the characteristic function of the unanimity game $\left(Z,\nu^Z\right)$. Accordingly, since $w_Y(X) > 0$ for all nonempty coalitions $X \subset Y \subset Z$, we have $\varphi_i\left(\nu^Z\right) > 0$ for all $i \in Z$. Furthermore, A1 implies that $\sum_{i\in Z}\varphi_i\left(\nu^Z\right) = 1$. As a result, we can identify the weight system w with a strictly positive sharing rule p such that $\varphi_i\left(\nu^Z\right) = p(i) > 0$ for all players $i \in Z$. Let $\left(Z,\nu^Y\right)$ be a unanimity game such that $Y \subset Z$ and $Y \neq Z$. The coalition Y being a p-type coalition for ν^Z, we have for any player $i \in Y$: $\varphi_i\left(\nu^Z\right) = \varphi_i(\varphi_Y\left(\nu^Z\right)\nu^Y)$. Using A3, this expression becomes $\varphi_i\left(\nu^Z\right) = \varphi_Y\left(\nu^Z\right) \times \varphi_i(\nu^Y)$, i.e.

$$\varphi_i(\nu^Y) = \frac{\varphi_i\left(\nu^Z\right)}{\varphi_Y\left(\nu^Z\right)} = \frac{p(i)}{p(Y)}.$$

Now, by Lemma 3, we have $\varphi_i(\nu^Y) = p_Y(i)$ and, then, the Luce choice axiom holds.

□

We are now able to establish the following result:

Theorem 4. : *For any TU-game (Z,ν), the Möbius value is a weighted value, i.e.*

$$\varphi_i(\nu) = \phi_i(\nu)$$

if and only if the sharing system $(Z,\mathcal{P})$ satisfies the Luce choice axiom.

Monderer and Samet (2002, Th. 5) have proved that a weighted value is a random order value that satisfies the partnership axiom, a result consistent with our Theorem 4. Hence, since a random order value is a Möbius value with a stochastically rationalizable sharing system (our Theorem 3), we know, using Luce and Suppes (1965), that the necessary and sufficient condition for the sharing system $(Z,\mathcal{P})$ to satisfy the Luce choice axiom is (1) to be stochastically rationalizable and (2) to satisfy the following condition:

$$\pi_Y^i = p_{Y_{+i}}(i) \times p_{Y_{+ij}}(j) \times p_{Y_{+ijk}}(k)\ldots$$

which always holds for weighted values.

Remark 5: Example 2 in Section 3.1 is associated with a sharing system that does not satisfy stochastic rationality (because $\gamma_\emptyset^2 = -1/6$) nor the Luce choice axiom (because $p^*(2) \neq p^*(23) \times p^*_{23}(2)$). Hence, it is neither a random order value nor a weighted value, but a Möbius value.

5 Properties of the Möbius Value

5.1 Monotone TU-Games

Most variations on the Shapley value assume that the positivity axiom holds: whenever the game is monotone, each individual value is positive (Monderer and Samet, 2002). Hence, the literature seems to focus on values for which the monotonicity of the game would be a sufficient condition for positivity. We show below that monotonicity is both a necessary and sufficient condition for any Möbius value to be positive. This implies that the positivity axiom may be replaced by the assumption of game monotonicity in the study of Möbius values.

Theorem 5. : *Any Möbius value $\varphi(\nu)$ is positive if and only if the TU-game (Z,ν) is monotone.*

Proof: It is sufficient to show that $\varphi_i(\nu) \geq 0$ for any player $i \in Z$. Using (10), we get:

$$\varphi_i(\nu) = \sum_{X \subset Z_{-i}} \underbrace{[\nu(X_{+i}) - \nu(X)]}_{(A)} \underbrace{\sum_{Y \supset X_{+i}} (-1)^{y-(x+1)} p_Y(i)}_{(B)}.$$

As (A) is positive if only if the game (Z,ν) is monotone, we just have to show that (B) is positive for each characteristic function ν, each player i and each coalition Y. Using Remark 2, we see that (B) corresponds to the definition of the Möbius inverse of a particular set function v^i, so that $p_{Y+i}(i)$ is the co-Möbius of that same function v^i. Now, we know that (i) a Möbius inverse is always nonnegative if and only if v^i is ∞-monotone, that is, v^i is a belief function and (ii) a characteristic function v^i is a belief function if and only if its co-Möbius is decreasing (see Shafer, 1976; Grabisch *et al.*, 2000). The individual sharing consistency condition shows that $p_Y(i)$ satisfies this last condition.

□

Since a quasivalue is defined by a solution characterized by the axioms A1-A3 as well as by positivity (Weber, 1988), it then follows from Theorem 1 that *a quasivalue is a Möbius value that satisfies the positivity axiom*. This proves our claim that quasivalues are special cases of Möbius values.

5.2 Convex TU-Games

We know from Shapley (1971) that the core of a convex game is nonempty. The following result shows that all the Möbius values belong to the core for a convex game.

Theorem 6. : *Any Möbius value $\varphi^p(\nu)$ is in the core of the TU-game (Z,ν) if and only if this game is convex.*

The proof is given in Appendix C.

We may now show that the set of Möbius values is identical to the core of a convex game. Indeed, when the game is convex, all the Möbius values belong to the core as shown by Theorem 6. Hence, for a nonconvex game, the set of random order values is a proper subset of Möbius values and, when the game is convex, we have the following result:

Theorem 7. : *For any TU-game (Z,ν), the set of all Möbius values is equal to its core if and only if the game is convex.*

Theorems 3 and 6 together with Weber's Theorem 14 imply that the core of a convex game being equal to the set of random order values, then the set of stochastically rationalizable Möbius values is equal to that of Möbius values, i.e. is equal to the core itself.

6 Concluding Remarks

Our approach to cooperative values allows us to shed new light on cooperative game theory. Indeed, we have shown that the weighted values correspond to the most constrained class of solutions. They are axiomatically characterized by Kalai and Samet (1987) through efficiency (A1), null-player (A2), additivity, positivity and partnership. Since positivity and additivity imply homogeneity (as shown by Kalai and Samet, 1987, p.213), the first two axioms may be replaced by linearity (A3) while positivity may be replaced by the stochastic rationality of the sharing system and partnership by the Luce choice axiom. Hence, our main results may be summarized as follows.

– For *any sharing system*, a solution that satisfies A1-A3 is *a Möbius value*.
– For *any stochastically rationalizable sharing system*, a solution that satisfies A1-A3 is *a random order value* (i.e. a solution that satisfies A1, A2, additivity and positivity).
– For *any sharing system satisfying the Luce choice axiom*, a solution that satisfies A1-A3 is *a weighted value* (i.e. a solution that satisfies A1, A2, additivity, positivity and partnership).

Some questions remain open. First, is there always an element in the nonempty core of a nonconvex game that can be represented by a Mobius value? If yes, what are the restrictions that the corresponding sharing system satisfies? And more generally, can Theorem 7 be extended to the case of nonconvex games with a nonempty core?

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Appendix A

It is useful to introduce the following three related concepts: a set function f, the Möbius inverse of f, denoted m, and the commonality function (Shafer, 1976) or co-Möbius inverse (Grabisch *et al.*, 2000) of f, denoted $\widehat{m}$. Then, the following four expressions simultaneously hold: for all $Y \in 2^Z_{-\emptyset}$,

$$\begin{cases} \widehat{m}(Y) = \sum_{X \supset Y} m(X) \\ m(Y) = \sum_{X \supset Y} (-1)^{x-y} \widehat{m}(X) \\ \widehat{m}(Y) = \sum_{X \supset \overline{Y}} (-1)^{z-x} f(X) \\ f(Y) = \sum_{X \supset \overline{Y}} (-1)^{x} \widehat{m}(X). \end{cases}$$

Appendix B

Proof of Theorem 2: The uniform Möbius value $\varphi(\nu, \mathcal{U})$ is defined for each nonempty coalition $X \subset Z$ by

$$\varphi_X(\nu, \mathcal{U}) = \sum_{\substack{Y \in 2^Z_{-\emptyset} \\ Y \supset X}} \Gamma_\nu(Y) u_Y(X) \tag{17}$$

where

$$u_Y(X) = \frac{x}{y},$$

x and y being the cardinalities of X and Y, respectively. Hence, by definition of the **PCC**, for each player $i \in Z$, (17) becomes

$$\begin{aligned} \varphi_i(\nu, \mathcal{U}) &= \sum_{Y \in 2^Z_{-\emptyset}} \Gamma_\nu(Y) u_Y(i) \\ &= \sum_{Y \in 2^Z_{-\emptyset}} \Gamma_\nu(Y) \frac{1}{y} \\ &= \sum_{\substack{Y \subset Z \\ i \in Y}} \sum_{X \subset Y} \frac{(-1)^{y-x} \nu(X)}{y} \\ &= \sum_{X \subset Z} \sum_{\substack{Y \subset Z \\ X_{+i} \subset Y}} \frac{(-1)^{y-x}}{y} \nu(X). \end{aligned} \tag{18}$$

Set

$$\lambda(i, X) \equiv \sum_{\substack{Y \subset Z \\ X_{+i} \subset Y}} \frac{(-1)^{y-x}}{y}.$$

When the player $i \in X$, there are $\binom{n-x}{y-x}$ coalitions Y such that $X \subset Y$. Consequently, we have:

$$\begin{aligned}
\lambda(i,X) &= \sum_{\substack{Y\subset Z \\ X_{+i}\subset Y}} \frac{(-1)^{y-x}}{y} \\
&= \sum_{y=x}^{n} (-1)^{y-x} \binom{n-x}{y-x} \frac{1}{y} \\
&= \sum_{y=x}^{n} (-1)^{y-x} \binom{n-x}{y-x} \int_0^1 t^{y-1} dt \\
&= \int_0^1 t^{x-1} \sum_{y=x}^{n} (-1)^{y-x} \binom{n-x}{y-x} t^{y-x} dt \\
&= \int_0^1 t^{x-1} (1-t)^{n-x} dt.
\end{aligned} \tag{19}$$

It is well known that

$$\int_0^1 t^{x-1} (1-t)^{n-x} dt = \frac{(x-1)!\,(n-x)!}{n!} = \lambda(i,X). \tag{20}$$

Note that, in (18), if the player $i \in X$, then $\lambda(i, X_{-i}) = -\lambda(i,X)$. Hence, (18) may be rewritten as follows:

$$\varphi_i(\nu, \mathcal{U}) = \sum_{\substack{X\subset Z \\ i\in X}} \lambda(i,X)\,(\nu(X) - \nu(X_{-i})). \tag{21}$$

Using (20) and (21), we then get the desired expression, i.e.

$$\varphi_i(\nu, \mathcal{U}) = \frac{1}{n!} \sum_{\substack{X\subset Z \\ i\in X}} (x-1)!\,(n-x)!\,(\nu(X) - \nu(X_{-i})) = S_i(\nu).$$

□

Appendix C

Proof of Theorem 6: (Sufficiency) If the TU-game (Z,ν) is convex, then we must show that $\sum_{i\in Y} \varphi_i^p(\nu) \geq \nu(Y)$ for all nonempty coalitions $Y \subset Z$, i.e. $\varphi_Y^p(\nu) \geq \nu(Y)$.

By (6), we know that:

$$\begin{aligned}
&\sum_{X\subset Y_{-\emptyset}} \sum_{T\subset \overline{Y}} \Gamma_\nu(X\cup T)\, p_{X\cup T}(X) \\
&= \sum_{X\subset Y_{-\emptyset}} \sum_{T\subset \overline{Y}} \sum_{S\subset T} (-1)^{t-s} \sum_{W\subset X} (-1)^{x-w} \nu(W\cup S)\, p_{X\cup T}(X)
\end{aligned} \tag{22}$$

whereas, by definition of a **PCC**,

$$\begin{aligned}
&\sum_{X\subset Y_{-\emptyset}} \sum_{T\subset Z\setminus Y} \sum_{S\subset T} (-1)^{t-s} \sum_{W\subset X} (-1)^{x-w} \nu(W)\, p_{X\cup T}(X) \\
&= \sum_{X\subset Y_{-\emptyset}} \Gamma_\nu(X)\, p_X(X) = \sum_{X\subset Y_{-\emptyset}} \Gamma_\nu(X) = \nu(Y).
\end{aligned} \tag{23}$$

Hence, from (22) and (23), we obtain:

$$\begin{aligned}\varphi_Y^p(\nu)-\nu(Y) &= \sum_{X\subset Y_{-\emptyset}}\sum_{T\subset\overline{Y}}\sum_{S\subset T}(-1)^{t-s}\sum_{W\subset X}(-1)^{x-w} \\ &\quad \times[\nu(W\cup S)-\nu(W)]\,p_{X\cup T}(X) \\ &= \sum_{X\subset Y_{-\emptyset}}\sum_{S\subset\overline{Y}}\sum_{W\subset X}(-1)^{x-w}[\nu(W\cup S)-\nu(W)] \\ &\quad \times\sum_{S\subset T\subset\overline{Y}}(-1)^{t-s}p_{X\cup T}(X) \\ &= \sum_{S\subset\overline{Y}}\sum_{R\subset\overline{Y}\setminus S}(-1)^{r}\sum_{X\subset Y_{-\emptyset}}\sum_{W\subset X}(-1)^{x-w} \\ &\quad \times[\nu(W\cup S)-\nu(W)]\,p_{X\cup S\cup R}(X) \\ &= \sum_{S\subset\overline{Y}}\sum_{R\subset\overline{Y}\setminus S}(-1)^{r} \\ &\quad \underbrace{\sum_{i\in Y}\sum_{\substack{X\subset Y\\ i\in X}}p_{X\cup S\cup R}(i)\times\sum_{W\subset X}(-1)^{x-w}[\nu(W\cup S)-\nu(W)]}_{(A)}.\end{aligned} \tag{24}$$

We may rewrite (A) as follows:

$$\sum_{X\subset Y_{-i}}\left\{\sum_{W\subset X_{+i}}(-1)^{(x+1)-w}[\nu(W\cup S)-\nu(W)]\right\} \times p_{X_{+i}\cup S\cup R}(i).$$

Hence, (A) is equivalent to:

$$\begin{aligned}&\sum_{X\subset Y_{-i}}\left\{\sum_{V\subset X}\sum_{W\subset V_{+i}}(-1)^{(v+1)-w}[\nu(W\cup S)-\nu(W)]\right\} \\ &\times\left\{\sum_{U\subset (Y_{-i})\setminus X}(-1)^{u}p_{U\cup X_{+i}\cup S\cup R}(i)\right\} \\ =&\sum_{\substack{X\subset Y\\ i\in X}}\underbrace{\sum_{\substack{V\subset X\\ i\in V}}\sum_{W\subset V}(-1)^{v-w}[\nu(W\cup S)-\nu(W)]}_{(B)} \\ &\times\left\{\sum_{U\subset Y\setminus X}(-1)^{u}p_{U\cup X\cup S\cup R}(i)\right\}.\end{aligned} \tag{25}$$

By interchanging the summations, (B) becomes

$$\begin{aligned}&\sum_{\substack{W\subset X\\ i\in W}}\left[\sum_{W\subset V\subset X}(-1)^{v-w}\right][\nu(W\cup S)-\nu(W)] \\ &+\sum_{W\subset X_{-i}}\left[\sum_{W\subset V\subset X_{-i}}(-1)^{v+1-w}\right][\nu(W\cup S)-\nu(W)] \\ =&\ \nu(X\cup S)-\nu(X)-\nu(X_{-i}\cup S)+\nu(X_{-i}).\end{aligned}$$

First, set

$$\sigma(i,X,S) \equiv [\nu(X\cup S)-\nu(X)]-[\nu(X_{-i}\cup S)-\nu(X_{-i})].$$

Since $X\cap S=\emptyset$, the convexity of (Z,ν) implies that $\sigma(i,X,S)\geq 0$. Second, setting $W\equiv U\cup R$, we have

$$\rho(i,X,S) \equiv \sum_{W\subset Z\setminus(X\cup S)} (-1)^{w} p_{X\cup S\cup W}(i).$$

Using the same argument as for (**??**), we obtain $\rho(i,X,S)\geq 0$.

Therefore, using (24) leads to

$$\varphi_Y^p(\nu)-\nu(Y)=\sum_{S\subset\overline{Y}}\sum_{i\in Y}\sum_{X\subset Y}\sigma(i,X,S)\times\rho(i,X,S)\geq 0.$$

(Necessity) The proof is by contradiction. Assume the TU-game (Z,ν) is not convex and show that there exists a Möbius value that does not belong to the core. First, applying Proposition 4 of Chateauneuf and Jaffray (1989) allows one to say that the **PCC** Γ_ν of ν satisfies:

$$\sum_{\{i,j\}\subset X\subset Y}\Gamma_\nu(X)\geq 0$$

for all pair of players $\{i,j\}$ belonging to each coalition $Y\subset Z$ if and only if the TU-game (Z,ν) is convex. Then, since our game is not convex, there exists a coalition $Y\subset Z$ and a pair of players $i,j\in Y$ such that:

$$\sum_{\{i,j\}\subset X\subset Y}\Gamma_\nu(X)<0. \tag{26}$$

We now have to prove that there exists a Möbius value, $\varphi^p(\nu)$, which is not in the core, that is, $\varphi^p_{Y_{-i}}(\nu)-\nu(Y_{-i})<0$. Recall that $p_X(Y_{-i})=1$ when $X\subset Y_{-i}$. From (6), it follows that:

$$\begin{aligned}\varphi^p_{Y_{-i}}(\nu)-\nu(Y_{-i}) &= \sum_{\substack{X\subset Z\\ X\not\subset Y_{-i}}}\Gamma_\nu(X)\,p_X(Y_{-i}) \\ &= \sum_{\substack{X\subset Y\\ i\in X}}\Gamma_\nu(X)\,p_X(Y_{-i}) \\ &\quad+\sum_{\substack{X\subset Z\\ X\not\subset Y}}\Gamma_\nu(X)\,p_X(Y_{-i}).\end{aligned} \tag{27}$$

Two cases may then arise. In the first one, we have $Y=Z$. Then, replace $\varphi^p(\nu)$ in (27) by $\varphi^{p_\varepsilon}(\nu)$ associated with the probability $p=p_\varepsilon$ in which $p_\varepsilon(i)=p_\varepsilon(j)=(1-\varepsilon)/2$ and $p_\varepsilon(k)=\varepsilon/(n-2)$ for all player $k\in Z_{-i}$. Then:

$$\lim_{\varepsilon\to 0}\left[\varphi^p_{Y_{-i}}(\nu)-\nu(Y_{-i})\right]=\frac{1}{2}\sum_{\{i,j\}\subset X\subset Y}\Gamma_\nu(X), \tag{28}$$

which is negative by (26), i.e., there exists a positive ε such that $\varphi^p_{Y_{-i}}(\nu)-\nu(Y_{-i})<0$.

In the second case, we have $Y\subsetneq Z$. Then, replace $\varphi^p(\nu)$ in (27) by $\varphi^{p_\varepsilon}(\nu)$ associated with the probability $p=p_\varepsilon$ where $p_\varepsilon(i)=p_\varepsilon(j)=\varepsilon$, $p_\varepsilon(k)=\varepsilon^2$ for all player $k\in Y_{-i}$ and

$$p_\varepsilon(k)=\frac{1-p_\varepsilon(Y)}{n-y}$$

for all player $k \in Z \backslash Y$. Again, (28) holds , i.e. there exists a positive ε such that $\varphi^{p}_{Y_{-i}}(\nu) - \nu(Y_{-i}) < 0$. Hence, if the TU-game (Z, ν) is not convex, the constructed Möbius value $\varphi^{p}(\nu)$ does not belong to the core.

$\square$

The RuBy Method for the Recommendation of a Best Choice From a Bipolar Valued Outranking Relation

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Abstract. This short note briefly presents a new and original way to solve the problem of a "best choice" recommendation in a multiple criteria decision aid framework. In particular it discusses how such a "best choice" can be constructed from a binary valued outranking relation defined on a finite set X of potential decision alternatives. The discussion is based on five natural principles.

1 Introduction

The goal of this extended abstract is to discuss how a "best choice"[3] recommendation may be rationally constructed from a binary valued outranking relation defined on a finite set X of potential decision alternatives. Such an outranking relation expresses the likelyhood of a global pairwise preference situation between the alternatives which combines an "at least as good" statement with the absence of any local veto. This decision aid problem is generally non trivial. In practise, most outranking relations result from a multiple criteria preference aggregation involving a majority concordance principle. In general such an aggregation doesn't produce a complete or transitive relation.

From a pragmatic point of view, the BC problematics is turned towards the selection of a unique ultimate "best" alternative. In practise, this kind of decision aid consists in the elicitation of a subset of "good" alternatives which is as restricted as possible. It is meant to help the decision maker to get as close as possible to the selection of a unique "best" alternative. In case this recommendation consists of several candidates, the decision aid process may be restarted with new and more detailed information in order to help selecting the final "best" alternative.

Apart from the European multiple criteria decision aid community [Roy85], this specific BC problematics has attracted quite low attention by the Operational Research field. Seminal work on it goes back to the first articles of Roy on the Electre I methods [Roy68,Roy69]. After Kitainik [Kit93], interest in solving the BC problem differently from the classical optimisation paradigm has reappeared. The recent work of Bisdorff and Roubens on valued kernels [BR96] has resulted in new attempts to solve the BC problem directly from the valued outranking graph. After first positive results [Bis00], methodological difficulties appeared when applying the outranking kernel concept to highly non transitive and partial outranking relations.

In this short note we therefore propose to present the major ideas of a new proposal to the BC problem and to revisit the logical and pragmatic foundations of this problematics. The objective is to propose a new and innovative decision aid methodology in the tradition of the pioneering work of Roy and Bouyssou [RB93].

[3] "best choice" will be written BC in the sequel.

2 Some fundamental concepts

Our starting point is a valued outranking digraph, denoted $\widetilde{G}^{\mathcal{L}}(X,\widetilde{\mathrm{S}})$, where X is a finite set of decision alternatives and $\widetilde{\mathrm{S}} : X \times X \to \mathcal{L}$ is a bipolar valued characterisation of an outranking relation on X taking its values in a bipolar evaluation domain $\mathcal{L}$.

Commonly $\mathcal{L}$ consists of the rational unit interval expressing the more or less credibility or robustness of an outranking statement. Throughout this paper we shall however suppose, except if stated otherwise, that $\mathcal{L} = \{-m,\ldots,0,\ldots,+m\}$ is a finite ordinal scale with $2m+1$ $(m \geq 1)$ values expressing a degree of likelyhood or robustness. If x and y are two alternatives of X, $\widetilde{\mathrm{S}}(x,y) = m$ signifies that the assertion "x outranks y" is *certainly true*; $\widetilde{\mathrm{S}}(x,y) > 0$ signifies that the assertion "x outranks y" is *more true than false*; $\widetilde{\mathrm{S}}(x,y) = 0$ signifies that the assertion "x outranks y" is *logically undetermined*, i.e. *neither true nor false*; $\widetilde{\mathrm{S}}(x,y) < 0$ signifies that the assertion "x outranks y" is *more false than true*; $\widetilde{\mathrm{S}}(x,y) = -m$ signifies that the assertion "x outranks y" is *certainly false*.

To be short we say that "x outranks y" is *$\mathcal{L}$-true* (respectively *$\mathcal{L}$-false*) if $\widetilde{\mathrm{S}}(x,y) > 0$ (respectively $\widetilde{\mathrm{S}}(x,y) < 0$).

A non empty subset Y of X is called a *choice* in $\widetilde{G}^{\mathcal{L}}$. Such a choice Y is said to be *$\mathcal{L}$-outranking* if and only if either, $Y = X$, or $x \notin Y \Rightarrow \exists y \in Y : \widetilde{\mathrm{S}}(y,x) > 0$. Similarly, a choice Y is said to be *$\mathcal{L}$-outranked* if and only if either $Y = X$, or $x \notin Y \Rightarrow \exists y \in Y : \widetilde{\mathrm{S}}(x,y) > 0$.

A choice Y is said to be *$\mathcal{L}$-independent* if and only if either, Y is a singleton, or $\forall x,y \in Y : \widetilde{\mathrm{S}}(x,y) < 0$. One should notice here that the concept of independence is not based on the negation of the $\mathcal{L}$-true outrankings. Such a negation would also include the couples of alternatives (x,y) for which $\widetilde{\mathrm{S}}(x,y) = 0$ holds.

An *$\mathcal{L}$-outranking* (*$\mathcal{L}$-outranked*) *kernel* is an $\mathcal{L}$-outranking ($\mathcal{L}$-outranked) and $\mathcal{L}$-independent choice.

The goal of our research is to determine a choice Y of X which can be used as a BC recommendation.

3 New foundations for the BC problematics

It is shown in [BRM05] that classical approaches to the BC problem present flaws and weaknesses. For example, the optimisation problem requires that any two alternatives are comparable. The Electre IS method [RB93] requires modifications of the original outranking digraph in order to present a single BC to the decision maker. The concept of $\mathcal{L}$-outranking kernel is also insufficient for the BC problematics, as it may not exist in certain digraphs or be only a subset of possible interesting recommendations.

Therefore we estimate that a new vision of this problem must be adopted. We define a new set of fundamental principles for the BC problematics. The two classical principles defined by Roy [Roy85] are still be present in this set, but are completed by 3 other *natural* ones.

A BC recommendation is a set of alternatives which will be used for a future proposal of a unique best alternative. This definition shows an important characteristic of any BC procedure. It should be interactive and tend towards the proposal of a unique best alternative. This observation is concordant with Roy's definition of the BC problem (see [Roy85]) where this implicit objective is emphasised.

A BC recommendation Y should therefore verify these 5 principles

- $\mathcal{B}_1$ Each alternative which in not selected must be considered as worse as at least one alternative of Y;
- $\mathcal{B}_2$ The subset of retained alternatives Y must be as small as possible;